

THE WAVE MECHANICS OF LARGE-SCALE STRUCTURE

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Abstract. I review the basic “gravitational instability” model for the growth of structure in the expanding Universe. This model requires the existence of small initial irregularities in the density of a largely uniform universe. These grow through linear and non-linear stages to form a complex network of clusters, filaments and voids. The dynamical equations describing the evolution of a self-gravitating fluid can be rewritten in the form of a Schrödinger equation coupled to a Poisson equation determining the gravitational potential. This approach has a number of interesting features, many of which were pointed out in a seminal paper by Widrow & Kaiser (1993). I argue that this approach has the potential to yield useful analytic insights into the dynamical growth of large-scale structure. As a particular example, I show that this approach yields an elegant reformulation of an idea due to Jones (1999) concerning the origin of lognormal intermittency in the galaxy distribution.

1. Introduction

The local Universe displays a rich hierarchical pattern of galaxy clustering that encompasses a large range of length scales, culminating in rich clusters and superclusters. The early Universe, however, was almost smooth, with only slight ripples seen in the cosmic microwave background radiation. Models of the evolution of structure link these observations through the effect of gravity, because the small initially overdense fluctuations accrete additional matter as the Universe expands. During the early stages, the ripples evolve independently, like linear waves on the surface of deep water,

but as the structures grow in mass, they interact with other in non-linear ways, more like nonlinear waves breaking in shallow water.

The linear theory of perturbation growth is well-established, but the non-linear regime is much more complicated and generally not amenable to analytic solution. Numerical N -body simulations have led the way towards an understanding of strongly developed clustering, but simulating a thing is not quite equivalent to understanding it. In this lecture, therefore, I sketch out some of the analytic methods that can be used to study non-linear clustering. I focus in particular on a novel approach based on the description of density fluctuations using quantum mechanics; see also Coles (2002).

2. Cosmological Structure Formation

The Big Bang theory is built upon the Cosmological Principle, a symmetry principle that requires the Universe on large scales to be both homogeneous and isotropic. Space-times consistent with this requirement can be described by the Robertson–Walker metric

$$ds_{\text{FRW}}^2 = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (1)$$

where κ is the spatial curvature, scaled so as to take the values 0 or ± 1 . The case $\kappa = 0$ represents flat space sections, and the other two cases are space sections of constant positive or negative curvature, respectively. The time coordinate t is called *cosmological proper time* and it is singled out as a preferred time coordinate by the property of spatial homogeneity. The quantity $a(t)$, the *cosmic scale factor*, describes the overall expansion of the universe as a function of time. If light emitted at time t_e is received by an observer at t_0 then the redshift z of the source is given by

$$1 + z = \frac{a(t_0)}{a(t_e)}. \quad (2)$$

The dynamics of an FRW universe are determined by the Einstein gravitational field equations which become

$$3 \left(\frac{\dot{a}}{a} \right)^2 = 8\pi G \rho - \frac{3\kappa c^2}{a^2} + \Lambda, \quad (3)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + 3\frac{p}{c^2} \right) + \frac{\Lambda}{3}, \quad (4)$$

$$\dot{\rho} = -3\frac{\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right). \quad (5)$$

These equations determine the time evolution of the cosmic scale factor $a(t)$ (the dots denote derivatives with respect to cosmological proper time

t) and therefore describe the global expansion or contraction of the universe. The behaviour of these models can further be parametrised in terms of the Hubble parameter $H = (\dot{a}/a)$ and the density parameter $\Omega = 8\pi G\rho/3H^2$, a suffix 0 representing the value of these quantities at the present epoch when $t = t_0$.

In order to understand how of large-scale structure arises, it is best to begin with the standard fluid-based approach to structure growth. In the standard treatment of the Jeans Instability one begins with the dynamical equations governing the behaviour of a self-gravitating fluid. These are the *Euler equation*

$$\frac{\partial(\mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \nabla\phi = 0 ; \quad (6)$$

the *continuity equation*

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0 , \quad (7)$$

expressing the conservation of matter; and the *Poisson equation*

$$\nabla^2\phi = 4\pi G\rho , \quad (8)$$

describing Newtonian gravity. If the length scale of the perturbations is smaller than the effective cosmological horizon $d_H = c/H$, a Newtonian treatment of cosmic structure formation is still expected to be valid in expanding world models. In an expanding cosmological background, the Newtonian equations governing the motion of gravitating particles can be written in terms of

$$\mathbf{x} \equiv \mathbf{r}/a(t) \quad (9)$$

(the comoving spatial coordinate, which is fixed for observers moving with the Hubble expansion),

$$\mathbf{v} \equiv \dot{\mathbf{r}} - H\mathbf{r} = a\dot{\mathbf{x}} \quad (10)$$

(the peculiar velocity field, representing departures of the matter motion from pure Hubble expansion), $\rho(\mathbf{x}, t)$ (the matter density), and $\phi(\mathbf{x}, t)$ (the peculiar Newtonian gravitational potential, i.e. the fluctuations in potential with respect to the homogeneous background) determined by the Poisson equation in the form

$$\nabla_{\mathbf{x}}^2\phi = 4\pi G a^2(\rho - \rho_0) = 4\pi G a^2\rho_0\delta. \quad (11)$$

In this equation and the following the suffix on ∇_x indicates derivatives with respect to the new comoving coordinates. Here ρ_0 is the mean background density, and

$$\delta \equiv \frac{\rho - \rho_0}{\rho_0} \quad (12)$$

is the *density contrast*. Using these variables the Euler equation becomes

$$\frac{\partial(a\mathbf{v})}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{1}{\rho}\nabla_{\mathbf{x}}p - \nabla_{\mathbf{x}}\phi. \quad (13)$$

The first term on the right-hand-side of equation (13) arises from pressure gradients, and is neglected in dust-dominated cosmologies. Pressure effects may nevertheless be important in the (collisional) baryonic component of the mass distribution when nonlinear structures eventually form. The second term on the right-hand side of equation (13) is the peculiar gravitational force, which can be written in terms of $\mathbf{g} = -\nabla_{\mathbf{x}}\phi/a$, the peculiar gravitational acceleration of the fluid element. If the velocity flow is irrotational, \mathbf{v} can be rewritten in terms of a velocity potential ϕ_v :

$$\mathbf{v} = -\nabla_{\mathbf{x}}\phi_v/a. \quad (14)$$

This is expected to be the case in the cosmological setting because (a) there are no sources of vorticity in these equations and (b) vortical perturbation modes decay with the expansion. We also have a revised form of the continuity equation:

$$\frac{\partial\rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_{\mathbf{x}}(\rho\mathbf{v}) = 0. \quad (15)$$

In order to understand how structures form we need to consider the difficult problem of dealing with the evolution of inhomogeneities in the expanding Universe. We are helped in this task by the fact that we expect such inhomogeneities to be of very small amplitude early on so we can adopt a kind of perturbative approach, at least for the early stages of the problem. The procedure is to linearise the Euler, continuity and Poisson equations by perturbing physical quantities defined as functions of Eulerian coordinates, i.e. relative to an unperturbed coordinate system. Expanding ρ , \mathbf{v} and ϕ perturbatively and keeping only the first-order terms in equations (13) and (15) gives the linearised continuity equation:

$$\frac{\partial\delta}{\partial t} = -\frac{1}{a}\nabla_{\mathbf{x}} \cdot \mathbf{v}, \quad (16)$$

which can be inverted, with a suitable choice of boundary conditions, to yield

$$\delta = -\frac{1}{aHf}(\nabla_{\mathbf{x}} \cdot \mathbf{v}). \quad (17)$$

The function $f \simeq \Omega_0^{0.6}$; this is simply a fitting formula to the full solution (Peebles 1980). The linearised Euler and Poisson equations are

$$\frac{\partial\mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} = -\frac{1}{\rho a}\nabla_{\mathbf{x}}p - \frac{1}{a}\nabla_{\mathbf{x}}\phi, \quad (18)$$

$$\nabla_{\mathbf{x}}^2 \phi = 4\pi G a^2 \rho_0 \delta; \quad (19)$$

$|\mathbf{v}|, |\phi|, |\delta| \ll 1$ in equations (17), (18) & (19). From these equations, and if one ignores pressure forces, it is easy to obtain an equation for the evolution of δ :

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega H^2 \delta = 0. \quad (20)$$

For a spatially flat universe dominated by pressureless matter, $\rho_0(t) = 1/6\pi G t^2$ and equation (20) admits two linearly independent power law solutions $\delta(\mathbf{x}, t) = D_{\pm}(t)\delta(\mathbf{x})$, where $D_+(t) \propto a(t) \propto t^{2/3}$ is the growing mode and $D_-(t) \propto t^{-1}$ is the decaying mode.

The above considerations apply to the evolution of a single Fourier mode of the density field $\delta(\mathbf{x}, t) = D_+(t)\delta(\mathbf{x})$. What is more likely to be relevant, however, is the case of a superposition of waves, resulting from some kind of stochastic process in which the density field consists of a superposition of such modes with different amplitudes. A statistical description of the initial perturbations is therefore required, and any comparison between theory and observations will also have to be statistical. Many versions of the inflationary scenario for the very early universe (Guth 1981; Guth & Pi 1982; Brandenberger 1985) predict the initial density fluctuations to take the form of a *Gaussian random field* in which the initial Fourier modes of the perturbation field have random phases.

3. Nonlinear Gravitational Instability

The linearised equations of motion provide an excellent description of gravitational instability at very early times when density fluctuations are still small ($\delta \ll 1$). The linear regime of gravitational instability breaks down when δ becomes comparable to unity, marking the commencement of the *quasi-linear* (or weakly non-linear) regime. During this regime the density contrast may remain small ($\delta < 1$), but the phases of the Fourier components $\delta_{\mathbf{k}}$ become substantially different from their initial values resulting in the gradual development of a non-Gaussian distribution function if the primordial density field was Gaussian. In this regime the shape of the power-spectrum changes by virtue of a complicated cross-talk between different wave-modes. Analytic methods are available for this kind of problem (Sahni & Coles 1995), but the usual approach is to use N -body experiments for strongly non-linear analyses (Davis et al. 1985; Jenkins et al. 1998).

Further into the non-linear regime, bound structures form. The baryonic content of these objects may then become important dynamically: hydrodynamical effects (e.g. shocks), star formation and heating and cooling of gas all come into play. The spatial distribution of galaxies may therefore be very different from the distribution of the (dark) matter, even on large

scales. Attempts are only just being made to model some of these processes with cosmological hydrodynamics codes, such as those based on Smoothed Particle Hydrodynamics (SPH; Monaghan 1992), but it is some measure of the difficulty of understanding the formation of galaxies and clusters that most studies have only just begun to attempt to include modelling the detailed physics of galaxy formation. In the front rank of theoretical efforts in this area are the so-called semi-analytical models which encode simple rules for the formation of stars within a framework of merger trees that allows the hierarchical nature of gravitational instability to be explicitly taken into account.

Perturbation theory fails when nonlinearities develop but it is important to stress that the fluid treatment is intrinsically approximate anyway. A fuller treatment of the problem requires a solution of the Boltzmann equation for the full phase-space distribution of the system $f(\mathbf{x}, \mathbf{v}, t)$ coupled to the Poisson equation (8) that determines the gravitational potential. In cases where the matter component is collisionless, the Boltzmann equation takes the form of a Vlasov equation:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^3 \left(\frac{\partial \phi}{\partial x_i} \frac{\partial f}{\partial v_i} - v_i \frac{\partial f}{\partial x_i} \right). \quad (21)$$

The fluid approach outline above can only describe cold material where the velocity dispersion of particles is negligible. But even if the dark matter is cold, there may be hot components of baryonic material whose behaviour needs also to be understood. Moreover, the fluid approach assumes the existence of a single fluid velocity at every spatial position. It therefore fails when orbits cross and multi-streaming generates a range of particle velocities through a given point.

Fortunately the formation of these structural elements can also be understood using simple models, especially that of Zel'dovich (1970). This approximation actually predicts that the density in certain regions – called *caustics* – should become infinite, but the gravitational acceleration caused by these regions remains finite. Of course, in any case one cannot justify ignoring pressure when the density becomes very high, for much the same reason as we discussed above in the context of spherical collapse: one forms shock waves which compress infalling material. At a certain point the process of accretion onto the caustic will stop: the condensed matter is contained by gravity within the final structure, while the matter which has not passed through the shock wave is held up by pressure. It has been calculated that about half the material inside the original fluctuation is reheated and compressed by the shock wave. An important property of the structures which thus form is that they are strongly unstable to fragmen-

tation. In principle, therefore, one can generate structure on smaller scales than the pancake.

I will now describe the Zel'dovich approximation in more detail, and show how it can follow the evolution of perturbations until the formation of pancakes. Imagine that we begin with a set of particles which are uniformly distributed in space. Let the initial (i.e. Lagrangian) coordinate of a particle in this unperturbed distribution be \mathbf{q} . Now each particle is subjected to a displacement corresponding to a density perturbation. In the Zel'dovich approximation the Eulerian coordinate of the particle at time t is

$$\mathbf{r}(t, \mathbf{q}) = a(t)[\mathbf{q} - b(t)\nabla_{\mathbf{q}}\Phi_0(\mathbf{q})], \quad (22)$$

where $\mathbf{r} = a(t)\mathbf{x}$, with \mathbf{x} a comoving coordinate, and we have made $a(t)$ dimensionless by dividing throughout by $a(t_i)$, where t_i is some reference time which we take to be the initial time. The derivative on the right hand side is taken with respect to the Lagrangian coordinates. The dimensionless function $b(t)$ describes the evolution of a perturbation in the linear regime, with the condition $b(t_i) = 0$, and therefore solves the equation

$$\ddot{b} + 2\left(\frac{\dot{a}}{a}\right)\dot{b} - 4\pi G\rho b = 0; \quad (23)$$

cf. equation (20). For a flat matter-dominated universe we have $b \propto t^{2/3}$ as before. The quantity $\Phi_0(\mathbf{q})$ is proportional to a velocity potential, of the type introduced above, i.e. a quantity of which the velocity field is the gradient:

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} - H\mathbf{r} = a\frac{d\mathbf{x}}{dt} = -a\dot{b}\nabla_{\mathbf{q}}\Phi_0(\mathbf{q}); \quad (24)$$

this means that the velocity field is irrotational. The quantity $\Phi_0(\mathbf{q})$ is related to the density perturbation in the linear regime by the relation

$$\delta = b\nabla_{\mathbf{q}}^2\Phi_0,$$

which is a simple consequence of Poisson's equation.

The Zel'dovich approximation is a linear approximation with respect to the particle displacements rather than the density, as was the linear solution we derived above. It is conventional to describe the Zel'dovich approximation as a kind of first-order Lagrangian perturbation theory, while what we have dealt with so far for $\delta(t)$ is a first order Eulerian theory. We have also assumed that the position and time dependence of the displacement between initial and final positions can be separated. Notice that particles in the Zel'dovich approximation execute a kind of inertial motion on straight line trajectories.

The Zel'dovich approximation, though simple, has a number of interesting properties. First, it is exact for the case of one dimensional perturbations up to the moment of shell crossing. As we have mentioned above, it also incorporates irrotational motion, which is required to be the case if it is generated only by the action of gravity (due to the Kelvin circulation theorem). For small displacements between \mathbf{r} and $a(t)\mathbf{q}$, one recovers the usual (Eulerian) linear regime: in fact, equation (22) defines a unique mapping between the coordinates \mathbf{q} and \mathbf{r} (as long as trajectories do not cross); this means that $\rho(\mathbf{r}, t)d^3r = \langle \rho(t_i) \rangle d^3q$ or

$$\rho(\mathbf{r}, t) = \frac{\langle \rho(t) \rangle}{|J(\mathbf{r}, t)|}, \quad (25)$$

where $|J(\mathbf{r}, t)|$ is the determinant of the Jacobian of the mapping between \mathbf{q} and \mathbf{r} : $\partial\mathbf{r}/\partial\mathbf{q}$. Since the flow is irrotational the matrix J is symmetric and can therefore be locally diagonalised. Hence

$$\rho(\mathbf{r}, t) = \langle \rho(t) \rangle \prod_{i=1}^3 [1 + b(t)\alpha_i(\mathbf{q})]^{-1} : \quad (26)$$

the quantities $1 + b(t)\alpha_i$ are the eigenvalues of the matrix J (the α_i are the eigenvalues of the deformation tensor). For times close to t_i , when $|b(t)\alpha_i| \ll 1$, equation (26) yields

$$\delta \simeq -(\alpha_1 + \alpha_2 + \alpha_3)b(t), \quad (27)$$

which is just the law of perturbation growth in the linear regime written a different way.

At some time t_{sc} , when $b(t_{sc}) = -1/\alpha_j$, an event called *shell-crossing* occurs such that a singularity appears and the density becomes formally infinite in a region where at least one of the eigenvalues (in this case α_j) is negative. This condition corresponds to the situation where two points with different Lagrangian coordinates end up at the same Eulerian coordinate. In other words, particle trajectories have crossed and the mapping between Lagrangian and Eulerian space is no longer unique. A region where the shell-crossing occurs is called a caustic. For a fluid element to be collapsing, at least one of the α_j must be negative. If more than one is negative, then collapse will occur first along the axis corresponding to the most negative eigenvalue. If there is no special symmetry, one therefore expects collapse to be generically one-dimensional, from three dimensions to two. Only if two (or three) negative eigenvalues, very improbably, are equal in magnitude can the collapse occur to a filament (or point). One therefore expects the formation of flattened structures to be the generic result of such collapse.

This is in accord with the classic work of Lin, Mestel & Shu (1965) who showed that, for a generic triaxial perturbation, the collapse is expected to occur not to a point, but to a flattened structure of quasi-two dimensional nature. The usual descriptive term for such features is *pancakes*.

The Zel’dovich approximation matches very well the evolution of density perturbations in full N -body calculations until the point where shell crossing occurs (Coles, Melott & Shandarin 1993). After this, the approximation breaks down completely. Particles continue to move through the caustic in the same direction as they did before. Particles entering a pancake from either side merely sail through it and pass out the opposite side. The pancake therefore appears only instantaneously and is rapidly smeared out. In reality, the matter in the caustic would feel the strong gravity there and be pulled back towards it before it could escape through the other side. Since the Zel’dovich approximation is only kinematic it does not account for these close-range forces and the behaviour in the strongly non-linear regime is therefore described very poorly. Furthermore, this approximation cannot describe the formation of shocks and phenomena associated with pressure.

Attempts to understand properties of non-linear structure using the fluid model therefore resort to further approximations (Sahni & Coles 1995) to extend the approach beyond shell-crossing. One relatively straightforward way to extend the Zel’dovich approximation is through the so-called *adhesion model* (Gurbatov, Saichev & Shandarin 1989). In this model one assumes that the particles stick to each other when they enter a caustic region because of an artificial viscosity which is intended to simulate the action of strong gravitational effects inside the overdensity forming there. This ‘sticking’ results in a cancellation of the component of the velocity of the particle perpendicular to the caustic. If the caustic is two-dimensional, the particles will move in its plane until they reach a one-dimensional interface between two such planes. This would then form a filament. Motion perpendicular to the filament would be cancelled, and the particles will flow along it until a point where two or more filaments intersect, thus forming a node. The smaller is the viscosity term, the thinner will be the sheets and filaments, and the more point-like will be the nodes. Outside these structures, the Zel’dovich approximation is still valid to high accuracy. Comparing simulations made within this approximation with full N -body calculations shows that it is quite accurate for overdensities up to $\delta \simeq 10$.

The spatial distribution of particles obtained using the adhesion approximation represents a sort of “skeleton” of the real structure: non-linear evolution generically leads to the formation of a quasi-cellular structure, which is a kind of “tessellation” of irregular polyhedra having pancakes for faces, filaments for edges and nodes at the vertices. This skeleton, however,

evolves continuously as structures merge and disrupt each other through tidal forces; gradually, as evolution proceeds, the characteristic scale of the structures increases. In order to interpret the observations we have already described, one can think of the giant “voids” as being the regions internal to the cells, while the cell nodes correspond to giant clusters of galaxies. While analytical methods, such as the adhesion model, are useful for mapping out the skeleton of structure formed during the non-linear phase, they are not adequate for describing the highly non-linear evolution within the densest clusters and superclusters. In particular, the adhesion model cannot be used to treat the process of merging and fragmentation of pancakes and filaments due to their own (local) gravitational instabilities, which must be done using full numerical computations.

4. An Alternative Approach

A novel approach to the study of collisionless matter, with applications to structure formation, was suggested by Widrow & Kaiser (1993). It involves re-writing of the fluid equations given in Section 2 in the form of a non-linear Schrödinger equation. The equivalence between this and the fluid approach has been known for some time; see Spiegel (1980) for historical comments. Originally the interest was to find a fluid interpretation of quantum mechanical effects, but in this context we shall use it to describe an entirely classical system.

To begin with, consider the continuity equation and Euler equation for a curl-free flow in which $\mathbf{v} = \nabla\phi$, in response to some general potential V . In this case the continuity equation can be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \phi) = 0 . \quad (28)$$

It is convenient to take the first integral of the Euler equation, giving

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 = -V , \quad (29)$$

which is usually known as the Bernoulli equation. The trick then is to make a transformation of the form

$$\psi = \alpha \exp(i\phi/\nu) , \quad (30)$$

where $\rho = \psi\psi^* = |\psi|^2 = \alpha^2$; the wavefunction $\psi(\mathbf{x}, t)$ evidently complex. Notice that the dimensions of ν are the same as ϕ , namely $[L^2 T^{-1}]$. After some algebra it emerges that the two equations above can be written in one equation of the form

$$i\nu \frac{\partial \psi}{\partial t} = -\frac{\nu^2}{2} \nabla^2 \psi + V\psi + P\psi , \quad (31)$$

where

$$P = \frac{\nu^2}{2} \frac{\nabla^2 \alpha}{\alpha}. \quad (32)$$

To accommodate gravity we need to couple equation (31) to the Poisson equation in the form

$$\nabla^2 \phi = 4\pi G \psi \psi^*, \quad (33)$$

taking V to be ϕ .

This system looks very similar to a Schrödinger equation, except for the extra “operator” P . The role of this term becomes clearer if one leaves it out of equation (31) and works backwards. The result is an extra term in the Bernoulli equation that resembles a pressure gradient. This is often called the “quantum pressure” that arises when one tries to understand a quantum system in terms of a classical fluid behaviour. Leaving it out to model a collisionless fluid can be justified only if α varies only slowly on the scales of interest. On the other hand one can model situations in which one wishes to model genuine effects of pressure by adjusting (or omitting) this term in the wave equation. Widrow & Kaiser (1993) advocated simply writing

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\phi(\mathbf{x})\psi, \quad (34)$$

i.e. simply ignoring the quantum pressure term. In this spirit, the constant \hbar is taken to be an adjustable parameter that controls the spatial resolution λ through a de Broglie relation $\lambda = \hbar/mv$. In terms of the parameter ν used above, we have $\nu = \hbar/m$, giving the correct dimensions for Planck’s constant. Note that the wavefunction ψ encodes the velocity part of phase space in its argument through the ansatz

$$\psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})} \exp[i\theta(\mathbf{x}/\hbar)], \quad (35)$$

where $\nabla\theta(\mathbf{x}) = \mathbf{p}(\mathbf{x})$, the local ‘momentum field’. This formalism thus yields an elegant description of both the density and velocity fields in a single function.

The approach outlined above is relatively new to galaxy clustering studies, and many details still need to be worked out. One source of complexity arises when one places the system in an expanding context. To see what happens, let us define a scaled density $\chi = \rho/\rho_0 = (1 + \delta)$ and take $\Omega = 1$. The continuity equation then becomes

$$\frac{\partial \chi}{\partial a} + \nabla \cdot (\chi \nabla \phi) = 0, \quad (36)$$

where the velocity potential ϕ is now such that $\mathbf{u} = d\mathbf{x}/dt = \nabla \phi$ and a is the scale factor. It is convenient to take the first integral of the Euler

equation, giving

$$\frac{\partial \phi}{\partial a} + \frac{1}{2}(\nabla \phi)^2 = -\frac{3}{2a}(\phi + \theta), \quad (37)$$

where $\theta = 2\Phi/3a^3H^2$ and Φ is the gravitational potential. After some more algebra the system again becomes a Schrödinger-like wave equation, but in a rather than in t and using $\psi^2 = \chi$, such that

$$i\nu \frac{\partial \psi}{\partial a} = -\frac{\nu^2}{2}\nabla^2 \psi + V\psi + P\psi, \quad (38)$$

with $V = \phi + \theta$ and P as before.

5. The Origin of Spatial Intermittency

Many types of non-linear system display a time-evolution characterized by the word “intermittency”. While linear Gaussian processes involve fluctuations that are symmetric about their mean value, non-linear processes typically have highly skewed distributions. In the context of time series, intermittent processes often have long quiescent periods punctuated by bursts of intense activity. In the spatial domain, intermittent processes are ones in which isolated regions of high density are separated by large voids; see Shandarin & Zel’dovich (1989).

One particular aspect of galaxy clustering that has received some attention over the years has been the property that the one-point probability distribution of density fluctuations $p(\rho)$ appears to have a roughly log-normal form, i.e. $\log \rho$ has a roughly normal distribution (Coles & Jones 1991). The lognormal is a prime distribution producing intermittency, and was discussed in a pioneering paper by Kolmogorov (1962). Although in a qualitative sense the application of the concept of intermittency to large-scale structure seems plausible, a quantitative description of how it arises is not easy to obtain. Drawing on ideas discussed by Zel’dovich et al. (1985, 1987), Jones (1999) suggested an analytical model for the cosmological context. On a simple level, this is quite easy to understand. If one has a linear process such that the output Y is constructed by co-adding a large number of independent contributing processes X_i ,

$$Y = \sum_{i=1}^N X_i \quad (39)$$

as $N \rightarrow \infty$ then the central limit theorem guarantees that Y is Gaussian as long as the X_i have finite variance. If one takes instead a multiplicative process of the form

$$Y = \prod_{i=1}^N X_i \quad (40)$$

with the X_i still independent then the same theorem suggests that $\log Y$ should be normal as $N \rightarrow \infty$. Lognormal distributions consequently arise naturally in random multiplicative processes, such as those involving fragmentation or coagulation.

Whatever the details of its origin, it is now established that this property has an interesting connection with the scaling properties of moments of the probability of the distribution. Taking a generic random variable X , such that the distribution of X within cells of side L is denoted $p(X; L)$, then the q -th moment at a given value of L is said to display scaling if

$$\langle X^q \rangle_L = \sum p(X; L) X^q \propto L^{\mu(q)}. \quad (41)$$

This means that different powers q of the density field vary as a different power of the coarse-graining scale L . The function $\mu(q)$ is called the *intermittency exponent*, and it can be extracted from observations. Jones, Coles & Martinez (1992) showed that observations suggest a roughly quadratic dependence of $\mu(q)$ upon q and that this is related to the underlying near-lognormal form of the density fluctuations. A set displaying the form (41) is usually termed a multifractal; see Paladin & Vulpiani (1987) for general discussion.

We know that the distribution of density fluctuations is not exactly lognormal. The intermittency exponent can be written in the form

$$\mu(q) = -(q - 1)D_q, \quad (42)$$

where the D_q are scaling dimensions (D_2 for example is the correlation dimension). We know that $D_q \propto q$ for a multifractal model whereas perturbative methods suggest a simpler form of scaling such that $D_q = D_0$ is constant, typical of a monofractal.

One aspect of this is that the hierarchy of correlation functions that describe a lognormal distribution display Kirkwood (1935) scaling, while it appears from numerical N -body simulations that cosmological fluctuations display a different hierarchical form. For a discussion of the relationship between lognormal and hierarchical scaling, see Coles & Frenk (1991).

It is within the overall framework of the fluid model that Jones (1999) sought to understand the observed intermittency of the large-scale structure of the Universe. Using the velocity potential introduced above, he first introduces an effective Bernoulli equation for the flow:

$$\frac{\partial \phi_v}{\partial t} - \frac{(\nabla \phi_v)^2}{2a^2} = \phi, \quad (43)$$

where ϕ is the actual gravitational potential. This equation neglects terms involving pressure gradients as mentioned above. To cope with shell-crossing

events, Jones (1999) introduces an artificial viscosity ν by adding a term to the right-hand-side of this equation:

$$\frac{\partial \phi_v}{\partial t} - \frac{(\nabla \phi_v)^2}{2a^2} = \phi + \frac{\nu}{a^2} \nabla^2 \phi_v. \quad (44)$$

The viscosity is introduced to prevent the particle flow from entering the multi-stream region by causing particles to stick together at shell-crossing. This is also used in an approach called the adhesion approximation (Gurbatov, Saichev & Shandarin 1989). Using the Hopf-Cole transformation $\phi_v = -2\nu \log \varphi$ and defining a scaled gravitational potential via $\phi = 2\nu \epsilon$ we can write the Bernoulli equation as

$$\frac{\partial \varphi}{\partial t} = \nu \nabla^2 \varphi + \epsilon(\mathbf{x}) \varphi. \quad (45)$$

This is called the random heat equation, because of the existence of the spatially-fluctuating potential term $\epsilon(\mathbf{x})$. The gravitational potential changes very slowly even during nonlinear evolution (Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994), so Jones (1999) assumes that it can be taken as constant and to be Gaussian distributed. An approximate scaling solution to (45) can then be found using a path integral adapted from that normally used in quantum physics (Feynman & Hibbs 1965); see below for more details. In this approximation, the function φ has a lognormal distribution (Coles & Jones 1991). We refer the reader to Jones (1999) for details; see also Zel'dovich et al. (1985, 1987).

This model is one of the few attempts that have been made to understand the non-linear behaviour of the matter distribution using analytic methods. Although not rigorous it surely captures the essential factors involved. It does, however, suffer from a number of shortcomings. First, the approach does not follow material beyond the shell-crossing stage. Second, the viscosity ν that is needed does not have properties that are very realistic physically: it can depend neither on the density ρ nor the position \mathbf{x} . Moreover, in the final analysis Jones (1999) takes the limit $\nu \rightarrow 0$, so it cancels out anyway. One is tempted to speculate that its introduction may be unnecessary. Third, the function $\varphi(\mathbf{x}, t)$ that emerges from equation (45) is not the desired density $\rho(\mathbf{x}, t)$ nor does it bear a straightforward relation to the density. Finally, it is not clear how the motion of a collisional baryonic component can be modelled within this framework.

This formulation provides a useful complementary approach to many techniques, including N -body methods. It also provides a new light with which to study the Jones (1999) model. Widrow & Kaiser (1993) show using theoretical arguments and numerical simulations that this system allows accurate numerical evolution of the system beyond shell-crossing, so

it does not have the ad hoc construction needed by the Jones (1999) model to remedy this.

Second, no artificial viscosity is required. Equations (45) and (34) are of the same form, apart from minor subtleties like the use of complex time coordinates. The potential term is easily understood in (34), and the wavefunction ψ now has a straightforward relationship to ρ . The upshot of this is that if one adopts the approximation of constant gravitational potential one can use the same path integral approach as described by Jones (1999). In a nutshell, given some initial wavefunction $\psi(\mathbf{x}', t')$ one can determine the wavefunction at a subsequent time $\psi(\mathbf{x}, t)$ using

$$\psi(\mathbf{x}, t) = \int K(\mathbf{x}, t; \mathbf{x}', t') \psi(\mathbf{x}', t') d^3 x', \quad (46)$$

where the function $K(\mathbf{x}, t; \mathbf{x}', t')$ involves a sum over all paths Γ connecting the initial and final states with $t > t'$:

$$K(\mathbf{x}, t; \mathbf{x}', t') = \int \mathcal{D}\Gamma \exp[iS(\Gamma)/\hbar], \quad (47)$$

where \mathcal{D} is an appropriate measure on the set of classical space-time trajectories. For a particle moving in a potential $V(\mathbf{x}, t) = m\phi(\mathbf{x}, t)$ the action S for a given path Γ is given by

$$S(\Gamma) = \int_{\Gamma} \left[\frac{1}{2} m \dot{x}^2 - m\phi(\mathbf{x}) \right] dt. \quad (48)$$

Note the presence of the Gaussian field in equation (48) and hence in the exponential of the integrand on the right-hand-side of equation (47). To get an approximate solution to this system we can follow the same reasoning as Zeldovich et al. (1985, 1987) and Jones (1999), ignoring time-varying terms, using the Gaussian properties and counting the dominant contributions to the path integral to deduce that the integral produces a solution of lognormal form. This part of the argument is identical to that advanced by Jones, except that the solution is for ψ rather than φ and since ρ is $|\psi|^2$ then one directly obtains a lognormal distribution for the desired density $\rho(\mathbf{x}, t)$.

It should be stressed that, although the present approach clearly provides a more elegant formulation of the problem, the deduction of lognormality remains approximate; the lognormal is not the exact solution to the system to either Jones' equation (45) or to equation (34). How accurately this approximate form applies is open to doubt and will have to be checked by full numerical solutions. Interestingly, however, it is known to apply quite accurately in quantum systems such as disordered mesoscopic electron configurations (Janssen 1998). As mentioned above, the Schrodinger

approach yields a wavefunction ψ which is directly related to the particle density ρ via $\rho = |\psi|^2$. Such quantum systems also display lognormal scaling for properties such as the conductance, which depends on $|\psi|^4$. It is a property of the lognormal distribution that if a random variable X is lognormal, then so is X^n . In such systems the role of the gravitational potential ϕ is played by a potential that describes the disorder of a solid, perhaps caused by the presence of defects. Such systems display *localisation* at low temperature which is similar in some ways to the original idea of Anderson localisation (Anderson 1958). The formation of strongly non-linear structures by gravity is thus directly analogous to the generation of localised wavefunctions in condensed matter systems.

Finally, and perhaps most promisingly for future work, the equation (34) offers a relatively straightforward way of modelling the behaviour of collisional material. The addition to the potential of a term of the form $\kappa|\psi|^2$ (with κ an appropriately-chosen constant), converts the original equation (34) into a nonlinear Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + m\phi(\mathbf{x})\psi + \kappa|\psi|^2\psi \quad (49)$$

(Sulem & Sulem 1999). This equation is now equivalent to those that describe the flow of a barotropic fluid; see Spiegel (1980). This system can therefore be used to model pressure effects, which are otherwise only handled effectively using numerical methods such as smoothed-particle hydrodynamical approximations (e.g. Monaghan 1992). In the context of quantum systems, the nonlinear term is used to describe the formation of Bose-Einstein condensates (e.g. Choi & Niu 1999).

6. Discussion

In this lecture I have sketched out an approach to the study evolving cosmological density fluctuations that relies on a transformation of the Vlasov-Poisson system into a Schrödinger-Poisson system. The transformation is not a new idea, but despite the efforts of Widrow & Kaiser (1993) it does not seem to be well known in the astronomical community. The immediate advantage of this new formalism is that it yields a rather more convincing approach to understanding the origin of spatial intermittency and approximate lognormality in the galaxy distribution than that offered by Jones (1999). It also makes a connection in the underlying physics with other systems that display similar phenomena.

On the other hand, one must be aware of the approximations also inherent in the present approach. The Schrödinger equation is not exact, and its usefulness as an approximate tool is restricted by a number of conditions

outlined by Widrow & Kaiser (1993); see also Spiegel (1980). Furthermore, the lognormal solution of the system is a further approximation and may not be valid especially in the strongly-fluctuating limit. Although it neatly bypasses some of the problems inherent in the Jones (1999) model, the nonlinear wave equation is by no means easy to solve in general situations. Numerical methods will still have to be employed to understanding other aspects of the evolution of cosmic structure within this framework as indeed they are in other branches of physics.

One particular issue worth exploring using this approach is to understand the limits of the approach in strongly non-linear situations. As it stands, the justification for the lognormal approximation arises from the weakly non-linear behaviour of collisionless matter moving in an almost constant potential field. Taking into account the expansion of the Universe, the changing gravitational potential, and the possible effects of matter pressure within in the action formalism may well reveal that a different form of hierarchical scaling pertains in the strongly non-linear regime.

It is perhaps worth mentioning some specific ideas of things that could be done using this approach and for which there seem to be clear benefits.

- **Perturbation Theory.** Standard perturbation methods do not guarantee a density field that is everywhere positive. Re-casting cosmological perturbation theory in terms of ψ , constructed so that $\psi^2 = \rho$ can remedy this.
- **Gas Pressure.** Analytic techniques for modelling the effects of gas pressure are scarce, even in relatively simple systems such as Lyman- α absorption cloud (Matarrese & Mohayee 2002). The quantum pressure term (36) or alternative terms such as in (56) allow flexibility to model gas behaviour at least at a phenomenological level.
- **Shell-crossing.** Methods such as the Zel'dovich approximation break down at shell-crossing, as described in Section 2.6. Although the simple *ansatz* I have used in this lecture does assume a unique velocity at every fluid location, it is possible to construct more complex representations that allow for multi-streaming (Widrow & Kaiser 1993). Note also that no singularities occur in the wavefunction at any time.
- **Reconstruction.** It is interesting to speculate that it might be possible to use the unitary structure of quantum mechanics in order to turn back the clock on evolved structure in order to reconstruct parts of the cosmic initial conditions.

In general many techniques exist for studying the wave mechanics of disordered systems, such as the renormalization group and path-integral methods, few of which are used by cosmologists working in this area. It is to be hoped that the introduction of some of these methods may allow

better physical insights into the behaviour of non-linear structure formation than can be found using brute-force N -body techniques.

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